Quantum phase space distributions in thermofield dynamics

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Abstract. It is shown that the quantum phase space distributions corresponding to a density operator ρ can be expressed, in thermofield dynamics, as overlaps between the state $|\rho\rangle$ and 'thermal' coherent states. The usefulness of this approach is brought out in the context of a master equation describing a nonlinear oscillator for which exact expressions for the quantum phase distributions for an arbitrary initial condition are derived.

1. Introduction

In quantum mechanics, the state of a system is represented by a density operator on a Hilbert space. The state can be pure or mixed. To each density operator one can associate several quantum phase space distributions which provide a quantum analogue of the classical phase space distribution. Prominent among these are the Wigner function [1], the Q-function [2] and the P-function [3] distribution each with its own special features. These three quantum phase space distributions belong to a one-parameter family of quantum phase space distributions introduced and investigated by Cahill and Glauber [4] and by Agarwal and Wolf [5, 6]. Over the years, quantum phase space distributions have not only proved to be useful computational tools by enabling one to transcribe operator equations into c-number language, but have also led to new concepts such as non-classical states of radiation. Recent developments in quantum state reconstruction [7], have made it possible to measure some of the quantum phase space distributions directly [8]. They are no longer auxiliary concepts useful only for computational purposes but have acquired a meaning in their own right. Several schemes for direct measurements of quantum phase space distributions [9] or the positivized versions thereof [10], as well as those in the context of atoms [11], have been proposed and are likely to be experimentally realized in the years to come.

   In this work, we examine the structure of the quantum phase space distributions from the point of view of thermofield dynamics [12, 13]. In the conventional formulation of quantum mechanics, pure states and mixed states are treated on an unequal footing. The formalism of thermofield dynamics overcomes this drawback by doubling the Hilbert space. In this formalism, density operators describing pure or mixed states are represented by a state vector in the doubled Hilbert space. The dynamical equations in both cases also acquire the structure of a Schrödinger equation even when dissipation is taken into account. The usefulness of this formalism for practical purposes can be seen from the exact algebraic solution [14] of a class

of master equations [15] describing coupled dissipative nonlinear oscillators. It is shown here that, in the framework of thermofield dynamics, quantum phase space distributions can be expressed as overlap between the state corresponding to the density operator and 'thermal' coherent states. To put it differently, quantum phase space distributions appear as coefficients of expansion when the state vector corresponding to a given density operator is expanded in terms of 'thermal' coherent states. That this aesthetically satisfying picture is also useful is demonstrated by giving an exact and straightforward algebraic treatment for the evolution of the quantum phase space distributions for the case of the master equation describing a dissipative nonlinear oscillator.

A brief summary of this work is as follows. In section 2 we briefly review the definition and properties of quantum phase space distributions. In section 3 we outline the formalism of thermofield dynamics and show how the statements in section 1 translate into the thermofield dynamics notation. Some illustrative examples of the computation of quantum phase space distributions are presented in section 4. Finally, in section 5 we apply the machinery developed in the preceding sections to obtain a complete picture of the time evolution of the phase space distributions corresponding to the density operators which evolve according to some standard master equations.

2. Quantum phase space distributions

In a series of papers Agarwal and Wolf have developed a general formulation of quantum phase space distributions. Crucial to their formulation is the notion of a Δ-operator. Here we shall confine ourselves to class of Δ-operators relevant for the present work. Consider the family of operators

$$\Delta^{(a)}(\alpha, \alpha^*) = \frac{1}{\pi} \int d^2\beta \, D(\beta) \, e^{(a-\frac{1}{2})|\beta|^2 - (\beta\alpha^* - \beta^*\alpha)} \qquad a \leqslant 1 \qquad (1)$$

where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$. These operators are Hermitian

$$\Delta^{(a)\dagger}(\alpha, \alpha^*) = \Delta^{(a)}(\alpha, \alpha^*) \qquad (2)$$

and have the following properties

$$\text{Tr}[\Delta^{(a)}(\alpha, \alpha^*)] = \pi \delta^2(\alpha) \qquad (3)$$

$$\text{Tr}[\Delta^{(a)}(\alpha, \alpha^*)\Delta^{(1-a)}(\beta, \beta^*)] = \pi \delta^2(\alpha - \beta) \qquad (4)$$

which follow from the fact that $\text{Tr}[D^\dagger(\alpha)D(\beta)] = \pi\delta^2(\alpha-\beta)$. With the help of these operators one can associate with a density operator ρ a class of quantum phase space distributions as follows:

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{1}{\pi} \text{Tr}[\rho \Delta^{(a)}(\alpha, \alpha^*)]. \qquad (5)$$

Conversely, ρ can be expressed in terms of these in the following manner

$$\rho = \int d^2\alpha \, \Phi_\rho^{(a)}(\alpha, \alpha^*)\Delta^{(1-a)}(\alpha, \alpha^*). \qquad (6)$$

For $a = 1, 1/2, 0$, $\Phi_\rho^{(a)}(\alpha, \alpha^*)$ respectively correspond to the P-function $P(\alpha, \alpha^*)$, the Wigner function $W(\alpha, \alpha^*)$ and the Q-function $Q(\alpha, \alpha^*)$. The family of quantum phase space distributions defined in this way are related to each other as follows:

$$\Phi_\rho^{(a-b)}(\alpha, \alpha^*) = \frac{1}{\pi b} \int d^2\beta \, \Phi_\rho^{(a)}(\beta, \beta^*) \exp\left[-\frac{|\alpha - \beta|^2}{b}\right] \qquad b \leqslant a \qquad (7)$$

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \exp\left[-(a-b)\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right]\Phi_\rho^{(b)}(\alpha, \alpha^*). \qquad (8)$$

It may be noted that the class of Δ-operators may also be expressed in the following manner

$$\Delta^{(a)}(\alpha, \alpha^*) = D(\alpha)\rho_0^{(a)} D^\dagger(\alpha) \tag{9}$$

where

$$\rho_0^{(a)} = \frac{1}{\pi} \int d^2\beta \, D(\beta) \, e^{(a-\frac{1}{2})|\beta|^2}. \tag{10}$$

The operator $\rho_0^{(a)}$ can be expressed in a more familiar form as

$$\rho_0^{(a)} = (1 - e^{-\theta}) \, e^{-\theta a^\dagger a} \qquad e^{-\theta} = -\frac{a}{1-a} \tag{11}$$

and is easily seen to have the structure of a 'thermal' vacuum. It corresponds to a genuine thermal vacuum only for $a \leqslant 0$. The family of quantum phase space distributions can therefore be compactly defined as follows:

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{1}{\pi} \mathrm{Tr}[\rho D(\alpha)\rho_0^{(a)} D^\dagger(\alpha)]. \tag{12}$$

This observation will be crucial to later developments. It may be remarked here that, expressed in this way, the quantum phase space distributions corresponding to a density operator ρ are identifiable as the generating function of the number distribution of the density operator $D^\dagger(\alpha)\rho D(\alpha)$:

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{1-\lambda}{\pi} \sum_n \lambda^n \langle n|D^\dagger(\alpha)\rho D(\alpha)|n\rangle \qquad \lambda = -a/(1-a). \tag{13}$$

That the family of quantum phase space distributions $\Phi_\rho^{(a)}(\alpha, \alpha^*)$ can be expressed as in (13) was, to our knowledge, first noticed by Moya-Cessa and Knight [16].

3. Thermofield dynamics

In thermofield dynamics (TFD) one associates with a density operator ρ acting on a Hilbert space $\mathcal{H}$, a state vector $|\rho^\alpha\rangle$, $1/2 \leqslant \alpha \leqslant 1$, in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}^*$ so that averages of operators with respect to ρ acquire the appearance of a scalar product:

$$\langle A \rangle = \mathrm{Tr} \, A\rho = \langle \rho^{1-\alpha}|A|\rho^\alpha \rangle. \tag{14}$$

The state $|\rho^\alpha\rangle$ is given by

$$|\rho^\alpha\rangle = \hat{\rho}^\alpha |I\rangle \tag{15}$$

where

$$\hat{\rho}^\alpha = \rho^\alpha \otimes I \tag{16}$$

and

$$|I\rangle = \sum |N\rangle \otimes |N\rangle \equiv \sum |N, N\rangle \tag{17}$$

where $|N\rangle$ constitute any complete orthonormal set in $\mathcal{H}$. The state $|I\rangle$ is simply the counterpart of the resolution of the identity

$$I = \sum |N\rangle\langle N| \tag{18}$$

in terms of a complete orthonormal set $|N\rangle$ in $\mathcal{H}$. In particular, if

$$\rho|N\rangle = p_N|N\rangle \tag{19}$$

$$|\rho^\alpha\rangle = \sum_N p_N^\alpha |N, N\rangle. \tag{20}$$

(It may be noted that for any density operator the states $|\rho^\alpha\rangle$, $1/2 \leqslant \alpha \leqslant 1$, have a finite norm in the extended Hilbert space. This is not necessarily so with the states $|\rho^{1-\alpha}\rangle$, $1/2 \leqslant \alpha \leqslant 1$, which include the state $|I\rangle$. These states are to be regarded as formal but extremely useful constructs.)

In this work we shall set $\alpha = 1$. This representation of the density operator as a state vector in the extended Hilbert space is the only useful one for discussing dissipative dynamics which will be of interest to us later. In this representation, for any Hermitian operator A, one has

$$\langle A \rangle = \mathrm{Tr}(A\rho) = \langle A|\rho\rangle. \tag{21}$$

Furthermore, for $|N\rangle$, we choose the number state $|n\rangle$ and introduce creation and annihilation operators $a^\dagger$, $\tilde{a}^\dagger$, a and $\tilde{a}$ as follows:

$$a|n,m\rangle = \sqrt{n}|n-1,m\rangle \qquad \tilde{a}|n,m\rangle = \sqrt{m}|n,m-1\rangle \tag{22}$$

$$a^\dagger|n,m\rangle = \sqrt{n+1}|n+1,m\rangle \qquad \tilde{a}^\dagger|n,m\rangle = \sqrt{m+1}|n,m+1\rangle. \tag{23}$$

The operators a and $a^\dagger$ commute with $\tilde{a}$ and $\tilde{a}^\dagger$. It is easily seen that the operators $\tilde{a}$ and $\tilde{a}^\dagger$ respectively simulate the action of $a^\dagger$ and a on $|n\rangle\langle m|$ from the right. From the expression for $|I\rangle$ in terms of the number states

$$|I\rangle = \sum_n |n,n\rangle \tag{24}$$

it follows that

$$a|I\rangle = \tilde{a}^\dagger|I\rangle \qquad a^\dagger|I\rangle = \tilde{a}|I\rangle \tag{25}$$

and hence for any operator

$$A(a^\dagger, a) = \sum_{p,q} \alpha_{p,q} a^{\dagger p} a^q \tag{26}$$

one has

$$A|I\rangle = \tilde{A}^\dagger|I\rangle \tag{27}$$

where $\tilde{A}$ is obtained from A by making the replacements (tilde conjugation rules) $a \to \tilde{a}$, $a^\dagger \to \tilde{a}^\dagger$, $\alpha \to \alpha^*$.

With the help of the formal constructs introduced above, it becomes possible to represent any density operator ρ as a state vector in the extended Hilbert space. Thus, for instance, the density operator for a coherent state $\rho = |\alpha\rangle\langle\alpha|$ can be represented as $|\rho\rangle = D(\alpha)\tilde{D}(\alpha^*)|0,0\rangle$. Similarly for the density operator corresponding to a thermal state one has

$$\rho = (1 - \mathrm{e}^{-\beta})\,\mathrm{e}^{-\beta a^\dagger a} \to |\rho\rangle = (1-f)\,\mathrm{e}^{fK_+}|0,0\rangle \tag{28}$$

where $f = \mathrm{e}^{-\beta}$ and $K_+ = a^\dagger \tilde{a}^\dagger$. The operator K_+ together with $K_- = a\tilde{a}$ and $K_3 = (a^\dagger a + \tilde{a}^\dagger \tilde{a} + 1)/2$ satisfies the algebra of $su(1,1)$

$$[K_-, K_+] = 2K_3 \qquad [K_3, K_\pm] = \pm K_\pm \tag{29}$$

with $K_0 = (a^\dagger a - \tilde{a}^\dagger \tilde{a})$ as the Casimir operator. Use of the disentangling theorem for $su(1,1)$ [14]

$$\exp(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-) = \exp(\Gamma_+ K_+)\exp((2\log\sqrt{\Gamma_3})K_3)\exp(\Gamma_- K_-) \tag{30}$$

where

$$\Gamma_\pm = \frac{2\gamma_\pm \sinh\phi}{2\phi\cosh\phi - \gamma_3\sinh\phi} \qquad \sqrt{\Gamma_3} = \left(\frac{2\phi}{2\phi\cosh\phi - \gamma_3\sinh\phi}\right) \tag{31}$$

with

$$\phi^2 = (\gamma_3^2/4) - \gamma_+\gamma_- \tag{32}$$

enables us to write the state $|\rho\rangle$ in (28) as

$$|\rho\rangle = e^{\bar{n}(K_+ + K_- - 2K_3)}|0, 0\rangle. \tag{33}$$

Turning to dynamics, consider, for example, the master equation for a nonlinear oscillator

$$\frac{\partial}{\partial t}\rho = -i[H, \rho] + \frac{1}{2}\gamma(\bar{n} + 1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \frac{1}{2}\gamma\bar{n}(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger) \tag{34}$$

where $H = \omega a^\dagger a + \chi(a^\dagger a)^2$. This master equation has been studied by a number of authors [15] in the context of nonlinear propagation in a Kerr medium. Applying $|I\rangle$ on (34) from the right and using (25), this master equation for ρ goes over to a Schrödinger-like equation for the state $|\rho\rangle$

$$\frac{\partial}{\partial t}|\rho\rangle = -i\hat{H}|\rho\rangle \tag{35}$$

where

$$-i\hat{H} = -i\omega(a^\dagger a - \tilde{a}^\dagger\tilde{a}) - i\chi[(a^\dagger a)^2 - (\tilde{a}^\dagger\tilde{a})^2] + \frac{1}{2}\gamma(\bar{n} + 1)(2a\tilde{a} - a^\dagger a - \tilde{a}^\dagger\tilde{a})$$
$$+ \frac{1}{2}\gamma\bar{n}(2a^\dagger\tilde{a}^\dagger - aa^\dagger - \tilde{a}\tilde{a}^\dagger). \tag{36}$$

In terms of the operators K_+, K_-, K_3 and K_0, the operator $-i\hat{H}$ can be written as

$$-i\hat{H} = -i(\omega - \chi)K_0 + \gamma(\bar{n} + 1)K_- + \gamma\bar{n}K_+ - (\gamma(2\bar{n} + 1) + 2i\chi K_0)K_3 + \frac{1}{2}\gamma \tag{37}$$

and hence the solution of (35) as

$$|\rho(t)\rangle = \exp(\gamma_0 K_0 + \frac{1}{2}\gamma t)\exp(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-)|\rho(0)\rangle \tag{38}$$

where

$$\gamma_+ = \gamma\bar{n}t \qquad \gamma_- = \gamma(\bar{n} + 1)t \qquad \gamma_3 = -(\gamma(2\bar{n} + 1) + 2i\chi K_0)t$$
$$\gamma_0 = -i(\omega - \chi)t. \tag{39}$$

Using the disentangling theorem (30), (38) can be written as

$$|\rho(t)\rangle = \exp(\gamma_0 K_0 + \frac{1}{2}\gamma t)\exp(\Gamma_+ K_+)\exp((2\log\sqrt{\Gamma_3})K_3)\exp(\Gamma_- K_-)|\rho(0)\rangle. \tag{40}$$

The fact that K_+, K_- and K_3 have simple actions on $|n, m\rangle$ enable one to solve (35) and hence (34) purely algebraically. Detailed expressions for $\rho_{m,n}(t)$ and the Q-function for an arbitrary initial condition may be found in [14].

In the context of the interaction of a single field mode with a non-thermal and phase insensitive environment, Agarwal [18] has considered the following master equation:

$$\frac{\partial}{\partial t}\rho = \kappa\left[a\rho a^\dagger + a^\dagger\rho a - \left(a^\dagger a + \frac{1}{2}\right)\rho - \rho\left(a^\dagger a + \frac{1}{2}\right)\right]. \tag{41}$$

In thermofield dynamics notation this translates into

$$\frac{\partial}{\partial t}|\rho\rangle = \kappa[K_+ + K_- - 2K_3]|\rho\rangle \tag{42}$$

so that

$$|\rho(t)\rangle = \exp(\kappa t(K_+ + K_- - 2K_3))|\rho(0)\rangle. \tag{43}$$

To conclude this section, we have seen how various density operators are represented in thermofield dynamics. We have also seen how certain master equations could be solved purely algebraically using the thermofield dynamics formalism. In the following section we shall show how the quantum phase space distributions are to be calculated in thermofield dynamics and how the time evolution of the quantum phase space distributions corresponding to the master equations considered here can be exactly determined.

4. Quantum phase space distributions in thermofield dynamics

From (12) and (21) it follows that

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{1}{\pi} \langle \alpha, \alpha^*; a|\rho\rangle \tag{44}$$

where

$$|\alpha, \alpha^*; a\rangle = \exp[-a[K_+ + K_- - 2K_z]]|\alpha, \alpha^*\rangle \tag{45}$$

$$= \frac{1}{1-a} D(\alpha) \tilde{D}(\alpha^*) \exp\left[-\frac{a}{1-a} K_+\right] |0, 0\rangle. \tag{46}$$

The quantum phase space distributions can thus be written as overlaps between the state $|\rho\rangle$ and 'thermal' coherent states. We now consider some examples to illustrate the usefulness of this formula for computation of quantum phase space distributions for some standard density operators.

(i) *Coherent states.* Consider the density operator $\rho = |\alpha_0\rangle\langle\alpha_0|$ for a coherent state. The corresponding state $|\rho\rangle$ is $|\alpha_0, \alpha_0^*\rangle$. Substituting this in (44) we obtain

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{1}{(1-a)\pi} \langle 0, 0| \exp\left[-\frac{a}{1-a} K_-\right] D(-\alpha) \tilde{D}(-\alpha^*)|\alpha_0, \alpha_0^*\rangle$$

$$= \frac{1}{(1-a)\pi} \langle 0, 0| \exp\left[-\frac{a}{1-a} K_-\right] |\alpha_0 - \alpha, \alpha_0^* - \alpha^*\rangle$$

$$= \frac{1}{(1-a)\pi} \exp\left[-\frac{a}{1-a}|\alpha - \alpha_0|^2\right] \langle 0, 0|\alpha_0 - \alpha, \alpha_0^* - \alpha^*\rangle$$

$$= \frac{1}{(1-a)\pi} \exp\left[-\frac{1}{1-a}|\alpha - \alpha_0|^2\right]. \tag{47}$$

In the limit $a \to 1$ one obtains the familiar expression for the P-function for the coherent state

$$\Phi_\rho^{(1)}(\alpha, \alpha^*) = \frac{1}{\pi} \langle \alpha, \alpha^*; 1|\rho\rangle = \delta^2(\alpha - \alpha_0). \tag{48}$$

Now, since

$$\langle \alpha, \alpha^*; a|\alpha_0, \alpha_0^*; b\rangle = \langle \alpha, \alpha^*; a + b|\alpha_0, \alpha_0^*\rangle \tag{49}$$

it follows that

$$\langle \alpha, \alpha^*; a|\alpha_0, \alpha_0^*; b\rangle = \begin{cases} \dfrac{1}{(1 - (a+b))} \exp\left[-\dfrac{1}{1 - (a+b)}|\alpha - \alpha_0|^2\right] & \text{if } a + b < 1 \\ \pi\delta^2(\alpha - \alpha_0) & \text{if } a + b = 1. \end{cases} \tag{50}$$

This relation will be used later.

(ii) *Squeezed thermal coherent state [19].* The density operator for a squeezed thermal coherent state is

$$\rho = (1 - e^{-\beta})D(\alpha_0)S(z)\exp(-\beta a^\dagger a)S^\dagger(z)D^\dagger(\alpha_0). \tag{51}$$

The corresponding $|\rho\rangle$ is given by

$$|\rho\rangle = (1 - f)D(\alpha_0)\tilde{D}(\alpha_0^*)S(z)\tilde{S}(z^*)\exp[fK_+]|0, 0\rangle \qquad f = e^{-\beta}. \tag{52}$$

Substituting this in (44) we obtain

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{(1 - f)(1 - \lambda)}{\pi} \langle 0, 0| e^{\lambda K_-} D^\dagger(\alpha)\tilde{D}^\dagger(\alpha^*)D(\alpha_0)\tilde{D}(\alpha_0^*)S(z)\tilde{S}(z^*) e^{fK_+}|0, 0\rangle. \tag{53}$$

By inserting the resolution of the identity in terms of two mode coherent states after $\exp(\lambda K_-)$ and before $\exp(f K_+)$, writing out the action of these operators on the coherent states and carrying out two of the four integrals that occur, we obtain

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{(1-f)(1-\lambda)}{\pi} \int \frac{\mathrm{d}^2\gamma}{\pi} \int \frac{\mathrm{d}^2\delta}{\pi} \, \mathrm{e}^{-\frac{1}{2}(1-\lambda^2)|\gamma|^2+\frac{1}{2}(1-\lambda)(\gamma\alpha^*-\gamma^*\alpha)-\frac{1}{2}(1-f^2)|\delta|^2}$$

$$\times \langle \gamma + \alpha - \alpha_0, \lambda\gamma^* + \alpha^* - \alpha_0^* | S(z)\tilde{S}(z*) | \delta, f\delta^* \rangle. \tag{54}$$

Finally, on using the known expression for the matrix element of the squeezing operator $S(z)$ between coherent states and carrying out the Gaussian integrals, we obtain

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{(1-f)(1-\lambda)\operatorname{sech} r}{\pi\sqrt{[(1-\lambda f)^2 - (\lambda - f)^2 \tanh^2 r]}} \exp\left[-\frac{(1-f)(1-\lambda)}{[(1-\lambda f)^2 - (\lambda - f)^2 \tanh^2 r]} X\right]$$

$$\tag{55}$$

where $\lambda = -a/(1-a)$ $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ and

$$X = [(1-\lambda f) - (\lambda - f)\tanh^2 r]|\alpha - \alpha_0|^2$$
$$-\tfrac{1}{2}(1+f)(1-\lambda)\tanh r[(\alpha - \alpha_0)^2 \,\mathrm{e}^{-\mathrm{i}\theta} + (\alpha^* - \alpha_0^*)^2 \,\mathrm{e}^{\mathrm{i}\theta}]. \tag{56}$$

In the limit of no squeezing, one recovers the familiar expressions for the quantum phase space distributions for a thermal coherent state

$$\Phi_\rho^{(a)}(\alpha, \alpha^*) = \frac{(1-f)}{(1-a(1-f))\pi} \exp\left[\frac{(1-f)}{1-a(1-f)}|\alpha - \alpha_0|^2\right]. \tag{57}$$

With this preparation, we are in a position to translate all the relations in section 2 into the language of thermofield dynamics.

An arbitrary state $|\rho\rangle$ can be in terms of $|\alpha, \alpha^*; a\rangle$ as follows:

$$|\rho\rangle = \int \mathrm{d}^2\beta \, \Phi_\rho^{(a)}(\beta, \beta^*)|\beta, \beta^*; 1-a\rangle. \tag{58}$$

This relation is the analogue of (6). The quantum phase space distributions thus acquire a new meaning as coefficients of expansion when $|\rho\rangle$ is expressed in terms of the 'thermal' coherent states $|\alpha, \alpha^*; 1-a\rangle$.

Taking the overlap of $|\rho\rangle$ as given in (58) with $|\alpha, \alpha^*; a-b\rangle$ and using (49) and (50) one obtains the relation (8) between quantum phase space distributions.

The relation (8) between the quantum phase space distributions comes about in the following way. The coherent state $|\alpha, \alpha^*\rangle$ has the following structure:

$$|\alpha, \alpha^*\rangle = \exp(\alpha(a^\dagger - \tilde{a}) - \alpha^*(a - \tilde{a}^\dagger))|0, 0\rangle. \tag{59}$$

The operators $(a^\dagger - \tilde{a})$ and $(a - \tilde{a}^\dagger))$ commute with each other. Now, from (59) it follows that

$$(a^\dagger - \tilde{a})|\alpha, \alpha^*\rangle = \frac{\partial}{\partial\alpha}|\alpha, \alpha^*\rangle \qquad (a - \tilde{a}^\dagger))|\alpha, \alpha^*\rangle = \frac{\partial}{\partial\alpha^*}|\alpha, \alpha^*\rangle \tag{60}$$

and hence

$$[K_+ + K_- - 2K_3]|\alpha, \alpha^*\rangle = -(a^\dagger - \tilde{a})(a - \tilde{a}^\dagger)|\alpha, \alpha^*\rangle = -\frac{\partial^2}{\partial\alpha\partial\alpha^*}|\alpha, \alpha^*\rangle. \tag{61}$$

The relation (8) therefore is a simple consequence of (44) and of the realization of $[K_+ + K_- - 2K_3]$ as a differential operator while acting on $|\alpha, \alpha^*\rangle$.

Finally, the representation (58) also leads to the relation

$$\mathrm{Tr}(A\rho) = \pi \int \mathrm{d}^2\beta \, \Phi_\rho^{(a)}(\beta, \beta^*)\Phi_A^{(1-a)}(\beta, \beta^*). \tag{62}$$

5. Time evolution of quantum phase space distributions

In section 3 we discussed some standard master equations and showed how they could be transcribed as Schrödinger-like equations. In the cases considered we showed that the solution of the Schrödinger-like equations thus obtained can be written as (38) with appropriate identification of the coefficients that occur there. In this section we wish to examine how transcription enables us to obtain a complete picture of the corresponding phase space distributions subject to given initial conditions. Two initial conditions are considered.

(i) *Evolution from an initial coherent state* $|\rho(0)\rangle = |\alpha_0, \alpha_0^*\rangle$. For this initial condition, it follows from (38), (44) and (45) that

$$
\begin{aligned}
\Phi^{(a)}_{\rho(t)}(\alpha, \alpha^*) &= \frac{1}{\pi} \langle \alpha, \alpha^*; a| \, e^{(\gamma_0 K_0 + \frac{1}{2}\gamma t)} \, e^{(\Gamma_+ K_+)} \, e^{((2\log\sqrt{\Gamma_3})K_3)} \, e^{(\Gamma_- K_-)} |\alpha_0, \alpha_0^*\rangle \\
&= \frac{1}{\pi} \, e^{\frac{1}{2}\gamma t} \langle \alpha, \alpha^*| \, e^{(-a(K_+ + K_- - 2K_3)} \, e^{(\Gamma_+ K_+)} \, e^{((2\log\sqrt{\Gamma_3})K_3)} \, e^{(\Gamma_- K_-)} \alpha_0 e^{-i\omega t}, \alpha_0^* e^{i\omega t}\rangle \\
&= \frac{1}{\pi} e^{\frac{1}{2}\gamma t} \langle \alpha, \alpha^*| \, e^{(\lambda K_+)} \, e^{((2\log(1-\lambda)K_3)} \, e^{(\lambda K_-)} \, e^{(\Gamma_+ K_+)} \, e^{((2\log\sqrt{\Gamma_3})K_3)} \, e^{(\Gamma_- K_-)}| \\
&\quad \times \alpha_0 \, e^{-i\omega t}, \alpha_0^* \, e^{i\omega t}\rangle
\end{aligned}
\tag{63}
$$

where $\lambda = -a/(1-a)$. Here in the last step we have used the disentangling theorem (30). Repeated use of the commutation relations and the disentangling theorem enable us to bring the right-hand side of (63) into the following standard form

$$
\Phi^{(a)}_{\rho(t)}(\alpha, \alpha^*) = \frac{1}{\pi} \langle \alpha, \alpha^*| \, e^{(\Gamma'_+ K_+)} \, e^{((2\log\sqrt{\Gamma'_3})K_3)} \, e^{(\Gamma'_- K_-)} |\alpha_0 \, e^{-i\omega t}, \alpha_0^* \, e^{i\omega t}\rangle
\tag{64}
$$

where

$$
\begin{aligned}
\Gamma'_+ &= 1 - \frac{[(1-\lambda)(1-\Gamma_+)]}{[1 - \lambda\Gamma_-]} \\
\Gamma'_- &= 1 - \frac{[(1-\Gamma_-)(1-\lambda\Gamma_+) - \lambda\Gamma_3]}{[1 - \lambda\Gamma_+]} \\
\sqrt{\Gamma'_3} &= \frac{[(1-\lambda)\sqrt{\Gamma_3}]}{[1 - \lambda\Gamma_+]}.
\end{aligned}
\tag{65}
$$

In this form the right-hand side of (64) can easily be evaluated. In the case of the nonlinear oscillator, $\Gamma_\pm$ and Γ_3 are functions of K_0 and (64) can be written as

$$
\Phi^{(a)}_{\rho(t)}(\alpha, \alpha^*) = \frac{1}{\pi} \sum_{m,n} \frac{1}{m!n!} (\alpha^*\alpha_0 \, e^{-i\omega t})^m (\alpha\alpha_0^* \, e^{i\omega t})^n (\sqrt{\Gamma_3})^{m+n+1} \, e^{(\Gamma'_+-1)|\alpha|^2} \, e^{(\Gamma'_--1)|\alpha_0|^2}
\tag{66}
$$

where it is understood that the K_0 in the expressions for $\Gamma_\pm$ and Γ_3 is replaced by $(m-n)$.

When $\chi = 0$, i.e. in the case of a linear oscillator, $\Gamma_\pm$ and Γ_3 are constants and (66) simplifies to

$$
\Phi^{(a)}_{\rho(t)}(\alpha, \alpha^*) = \frac{\sqrt{\Gamma'_3}}{\pi} \, e^{\frac{1}{2}\gamma t} \, e^{(\Gamma'_+-1)|\alpha|^2} \, e^{(\Gamma'_--1)|\alpha_0|^2} \, e^{\sqrt{\Gamma'_3}(\alpha^*\alpha_0 e^{-i\omega t} + \alpha\alpha_0^* e^{i\omega t})}.
\tag{67}
$$

(ii) *Evolution from a given initial quantum phase space distribution.* An arbitrary initial state can be expanded as

$$
|\rho(0)\rangle = \int d^2\alpha_0 \, \Phi^{(a)}_{\rho(0)}(\alpha_0, \alpha_0^*) |\alpha_0, \alpha_0^*; 1-a\rangle.
\tag{68}
$$

For the time evolution of the quantum phase space distributions one therefore obtains

$$\Phi_{\rho(t)}^{(a)}(\alpha, \alpha^*) = \int d^2\alpha_0 \, K^{(a)}(\alpha, \alpha^*, t; \alpha_0, \alpha_0^*, 0)\Phi_{\rho(0)}^{(a)}(\alpha_0, \alpha_0^*) \tag{69}$$

where

$$K^{(a)}(\alpha, \alpha^*, t; \alpha_0, \alpha_0^*, 0) = \frac{1}{\pi} \langle \alpha, \alpha^*; a| e^{(\gamma_0 K_0 + \frac{1}{2}\gamma t)} e^{(\gamma_+ K_+ + \gamma_3 K_3 + \gamma_- K_-)} |\alpha_0, \alpha_0^*; (1-a)\rangle. \tag{70}$$

Using the same manipulations as above, one finds that

$$K^{(a)}(\alpha, \alpha^*, t; \alpha_0, \alpha_0^*, 0) = \frac{e^{\gamma t/2}}{\pi} \langle \alpha, \alpha^*| e^{(\Gamma_+'' K_+)} e^{((2\log\sqrt{\Gamma_3''})K_3)} e^{(\Gamma_-'' K_-)} |\alpha_0 e^{-i\omega t}, \alpha_0^* e^{i\omega t}\rangle \tag{71}$$

where

$$\Gamma_+'' = 1 - (1 - \lambda)\frac{[(1 - \Gamma_+)(\lambda - \Gamma_-) - \Gamma_3]}{[(\lambda - \Gamma_-)(1 - \lambda\Gamma_+) - \lambda\Gamma_3]}$$

$$\Gamma_-'' = 1 + (1 - \lambda)\frac{[(1 - \Gamma_-)(1 - \lambda\Gamma_+) - \lambda\Gamma_3]}{[(\lambda - \Gamma_-)(1 - \lambda\Gamma_+) - \lambda\Gamma_3]}$$

$$\sqrt{\Gamma_3''} = -\frac{(1 - \lambda)^2\sqrt{\Gamma_3}}{[(\lambda - \Gamma_-)(1 - \lambda\Gamma_+) - \lambda\Gamma_3]}. \tag{72}$$

The right-hand side of (71) can be cast into the form (67) for the case of the linear oscillator and in the form (66) in the case of the nonlinear oscillator.

Finally, for the master equation (41), owing to its structure, one has the following interesting result

$$\Phi_{\rho(t)}^{(a)}(\alpha, \alpha^*) = \Phi_{\rho(0)}^{(a-\kappa t)}(\alpha, \alpha^*) \tag{73}$$

which says that $\Phi_{\rho(t)}^{(a)}(\alpha, \alpha^*)$ sweeps through the entire family of the quantum phase space distributions $\Phi_{\rho(0)}^{(b)}(\alpha, \alpha^*)$, $b \leqslant a$, associated with the initial density operator.

6. Conclusion

The observation that the family of quantum phase space distributions considered by Agarwal and Wolf have the structure as in (12) and the recognition that this form, in thermofield dynamics, translates into an overlap as in (44) constitute the key results of this work. These results, in turn, enable us not only to understand various relations between quantum phase space distribution in an aesthetically satisfying manner, but also prove to be extremely useful for computational purposes. The latter aspect of the formalism developed here is demonstrated by using it to obtain a full time-dependent solution for a family of quantum phase space distributions for the master equation describing a nonlinear dissipative oscillator. We hope that the results presented here will be useful in the context of wavepacket dynamics in nonlinear systems in the presence of dissipation, a subject which has attracted considerable attention in recent years.

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